Exchange Graphs of Maximal Weakly Separated Collections

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PRIMES Conference, May 16th, 2015

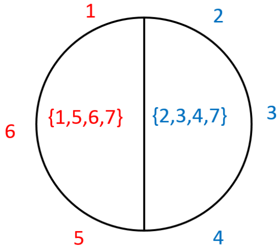
Weakly Separated Sets in $[n]$

$[n] = \{1, 2..., n\}$

Definition

Two subsets $A, B \subset [n]$ of the same cardinality are called **weakly separated** if $A \setminus B$ and $B \setminus A$ can be separated by a chord in the circle.

Example: $\{1, 5, 6, 7\}$ and $\{2, 3, 4, 7\}$

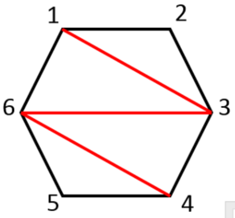


Weakly Separated Collections: The Simplest Case - Triangulations

$\binom{[n]}{k}$ = set of k-element subsets of $[n]$

Basic idea: A collection $S \subset \binom{[n]}{k}$ is called **weakly separated** if the subsets in S are pairwise weakly separated. Simplest case: $k = 2$

Example: $n = 6$



{1,3}, {3, 6}, {4, 6},
{1,2}, {2,3}, {3,4}, {4, 5},
{5, 6}, {6, 1}

$\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{1,6\}$ are in all triangulations.

Grassmann Necklace and Positroids

Definition (Oh, Speyer, Postnikov)

A connected **Grassmann Necklace** $\mathcal{I}$ is a collection of distinct sets $I_1, I_2, \ldots I_n \in \binom{[n]}{k}$ such that $I_{i+1} \supset I_i \setminus i$ where indices are considered modulo n.

Example: Hexagon
$\mathcal{I} = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{1, 6\}.$

Definition

The **positroid** $\mathcal{M}_{\mathcal{I}}$ is a subset of $\binom{[n]}{k}$ that is a function of the Grassmann necklace $\mathcal{I}$.

Example: Hexagon
$\mathcal{M}_{\mathcal{I}} = \binom{[6]}{2}.$

Maximal Weakly Separated Collections

Definition

A weakly separated collection S is said **to have Grassmann Necklace** $\mathcal{I}$ if $\mathcal{S} \subset \mathcal{M}_{\mathcal{I}} \subset \binom{[n]}{k}$. S is said to be **maximal** if it is not contained in another weakly separated collection with the same Grassmann necklace.

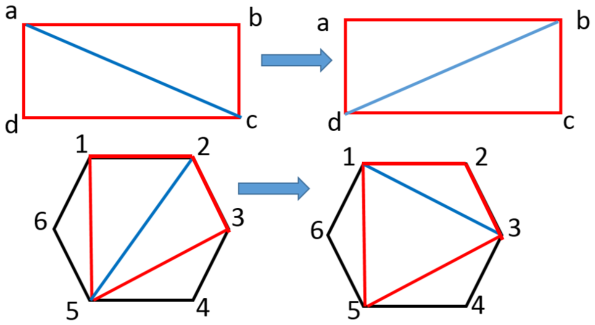
Theorem (Oh, Postnikov, Speyer)

The cardinality of a maximal weakly separated collection with Grassmann Necklace $\mathcal{I}$ *is a fixed number that is a function of the Grassmann Necklace.*

For triangulations, the cardinality is $2n - 3$.

Mutations in the Case of Triangulations

In the case of a triangulation, a mutation corresponds to a diagonal flip.
$\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, c\}\}$ mutates to $\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{b, d\}\}$.



Mutations

Definition (Oh, Postnikov, Speyer)

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a maximal weakly separated collection with Grassmann Necklace $\mathcal{I}$. For a $k-2$ element subset H of $[n]$, suppose there exist k-element subsets $H \cup \{a,b\}, H \cup \{b,c\}, H \cup \{c,d\}, H \cup \{a,d\}, H \cup \{a,c\} \in F$. Then we can obtain the maximal weakly separated collection $\mathcal{F}' = (\mathcal{F} \setminus \{H \cup \{a,c\}\}) \cup \{H \cup \{b,d\}\}$ through a mutation.

Example:
$\mathcal{I} = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{1,4,5\},\{1,2,5\}\}$.
$\{\{2,3,4\},\{3,4,5\},\{1,2,3\},\{1,3,4\},\{1,4,5\},\{1,2,5\},\{1,3,5\}\}$
$\{\{2,3,4\},\{3,4,5\},\{1,2,3\},\{1,3,4\},\{1,4,5\},\{1,2,5\},\{1,2,4\}\}$

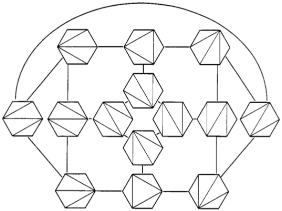
Exchange Graphs

Definition

Let $\mathcal{I}$ be a Grassmann necklace. The **exchange graph** $\mathcal{G}^{\mathcal{I}}$ is defined as follows: the vertices are all of the maximal weakly separated collections with Grassmann Necklace $\mathcal{I}$ and V_1 and V_2 form an edge if V_1 can be mutated into V_2.

Example: Hexagon

Let $\mathcal{I} = \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{1,6\}$.



Known Result about Exchange Graphs

Theorem (Oh, Postnikov, Speyer)

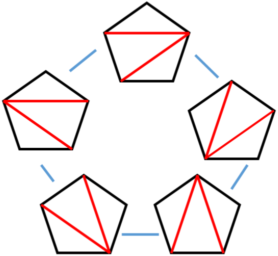
The exchange graph is connected.

Cycles

Theorem

If an exchange graph $\mathcal{G}^{\mathcal{I}}$ is a single cycle, then it must have length 1, 2, 4, or 5. We can construct exchange graphs for all of these lengths.

Example: Pentagon



Trees

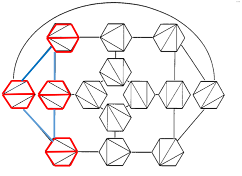
Theorem

If an exchange graph $\mathcal{G}^{\mathcal{I}}$ is a tree, then it must be a path. For any $k \geq 0$, there exists an exchange graph $\mathcal{G}^{\mathcal{I}}$ that is a path of length k.

Example: Quadrilateral

C-Constant Graphs: Certain Subgraphs of The Exchange Graph

C-constant graphs are a generalization of exchange graphs.
Example:



Definition

Given a weakly separated collection $C \subset \mathcal{M}_{\mathcal{I}}$ such that $\mathcal{I} \subset \mathcal{C}$, we define the **C-constant graph** $\mathcal{G}^{\mathcal{I}}(C)$ to be the vertex-induced subgraph of $\mathcal{G}^{\mathcal{I}}$ generated by all maximal weakly separated collections containing C.

Known Result about C-Constant Graphs

Theorem (Oh, Speyer)

The C-constant graph is connected.

Connection between C-Constant Graphs and Exchange Graphs

We found a nice isomorphism that links the C-constant graphs with the exchange graphs, thus showing that the C-constant graphs are not actually more general:

Theorem

For any $c \geq 0$, the set of possible C-constant graphs of co-dimension c is isomorphic to the set of the possible exchange graphs $\mathcal{G}^{\mathcal{I}}$ with interior size c (for $S \in \mathcal{G}^{\mathcal{I}}$, $|S| - |I| = c$.)

Construction of Certain C-Constant Graphs: Cartesian Product

We found a nice way to construct certain "bigger" exchange graphs of higher co-dimension from "smaller exchange graphs of lower co-dimension:

Theorem

For $\mathcal{G}^{\mathcal{I}}(C)$ with co-dimension c and $\mathcal{G}^{\mathcal{J}}(D)$ with co-dimension d, there exists an E-constant graph with co-dimension $c + d$ isomorphic to the Cartesian product $\mathcal{G}^{\mathcal{I}}(C) \square \mathcal{G}^{\mathcal{J}}(D)$.

Characterization of Possible Orders of C-Constant Graphs of Low Co-Dimension

Oh and Speyer characterized the possible orders of C-Constant graphs of co-dimensions 0 and 1. We extended this result to co-dimension 2,3, and 4.

Maximum orders of C-Constant Graphs of a fixed co-dimension c:

$c = 0 : 1$

$c = 1 : 2$

$c = 2 : 5$

$c = 3 : 14$

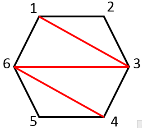
$c = 4 : 42$

Catalan Conjecture

Observation

Let $M(c)$ be the maximum possible order of C-constant graph of co-dimension c. For any $c \geq 0$, a lower bound on $M(c)$ is $c+1$'th Catalan number.

Construction: Exchange graph of a $c+3$-gon (Interior size $= c$)



Interior Size $= 3$, $|\mathcal{G}(C)| = 14$.

Conjecture

For any $c \geq 0$, $M(c) = C_{c+1}$. We've proven this for $c \leq 4$.

Future Work

For a fixed co-dimension c, determine the possible sizes of $\mathcal{G}^{\mathcal{I}}(C)$ in terms of c.

Try to prove the Catalan Conjecture.

Try to build C-constant graphs with co-dimension c from C-constant graphs with smaller co-dimension.

Continue to work on characterizing associated decorated permutations for various classes of exchange graphs.

Thank You

Thanks to:
My mentor Miriam Farber
Professor Postnikov for suggesting the project
My parents
and the MIT PRIMES Program.